Discontinuous Galerkin method for macroscopic traffic flow models on networks using numerical fluxes at junctions

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Introduction

Modelling of traffic flows will have an important role in the future. With a rising number of cars on the roads, we must optimize the traffic situation. That is the reason we started to study traffic flows. It is important to have working models which can help us to improve traffic flow. We can model real traffic situations and optimize e.g. the timing of traffic lights or local changes in the speed limit. The benefits of modelling and optimization of traffic flows are both ecological and economical.

Methodology

Our work [1] describes the numerical solution of traffic flows on networks. We solve especially the macroscopic models. This approach considers traffic as a continuum which flows through the roads. Using these models, it is possible to make simulations on big networks with a large number of cars. These models are described by partial differential equations in the form of conservation laws:

$$\frac{\partial}{\partial t}\rho\left(x,t\right)+\frac{\partial}{\partial x}Q\left(x,t\right)=0, \tag{1}$$

where $\rho\left(x,t\right)$ and $Q\left(x,t\right)$ are the unknown traffic density and traffic flow at position x and time t, respectively. We have only one equation (1) for two unknowns. Thus, we use the Lighthill–Whitham–Richards model where $Q\left(x,t\right)$ is taken as the equilibrium velocity $Q_e(\rho(x,t))$, cf. [1].

Due to the character of equation (1), we can expect discontinuity of the traffic density $\rho\left(x,t\right)$. Therefore, for the numerical solution of our models, we use the discontinuous Galerkin (DG) method in space and a forward Euler method in time. We introduced limiters which prevent spurious oscillations in the numerical solution and keep density in an admissible interval. All the above was performed on networks. Thus, we had to deal with the problem of boundary conditions at the junctions.

In [1] we introduce our own approach to boundary conditions at junctions, which uses special numerical flux choices. This approach is new and the behavior of the resulting model can be interpreted as the introduction of turning lanes in front of the junction. This is a different approach to the models in [2] and [3], which correspond to single-lane roads where overtaking is prohibited. Moreover, the presented construction of the traffic flux at junctions allows the simulation of arbitrary traffic light combinations instead of only full red/green lights as in [2] and [3].

We prove several important properties of our proposed numerical scheme, such as a discrete analogue to the Rankine–Hugoniot conditions for the numerical fluxes at the junction, conservation property of the DG scheme and traffic distribution error, cf. [1, Lemma 2, Theorems 1 and 2].

Results

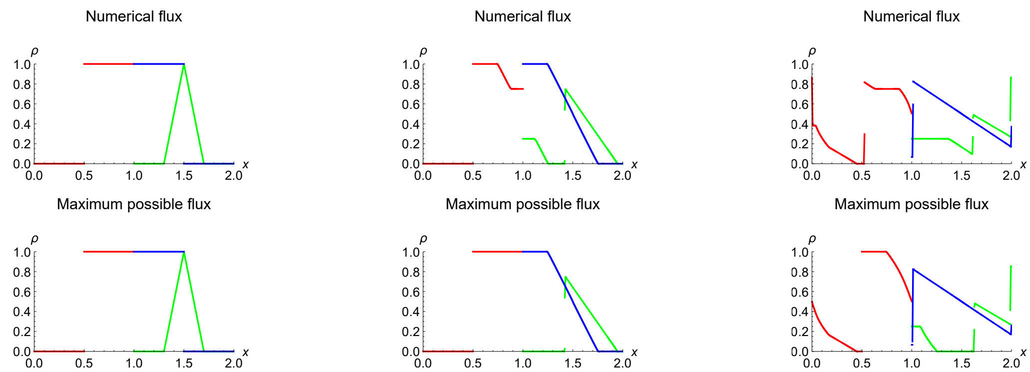
We demonstrate the main properties of our approach with a simple numerical experiment with junctions. We can compare our result with the results obtained by the approach in the paper [3].

We define the simple network with three roads and two junctions. This network forms a closed loop, so the total number of cars is conserved. The length of all roads is normed to 1. At the first junction we have one incoming road (red) and two outgoing roads (green and blue). At the second junction we have the opposite situation – roads 2 and 3 merge to road 1. We consider a different distribution of cars to the two outgoing roads at the first junction: $\frac{3}{4}$ go from the first road to the second and $\frac{1}{4}$ from the first road to the third. The initial condition is shown in Figure 1a.

We compare our approach with that of [3] which uses the maximum possible flux through a junction while satisfying the preferred traffic distribution. In both approaches we use the Godunov numerical flux and the forward Euler method. A right of way parameter q must be prescribed for the junction with two incoming roads in the case of the maximum possible flux. We use $q = 0.5$, so the roads are equal. In our approach, we do not have a defined right of way, so the roads are equal as well.

We can see the comparison in Figure 1. Our approach is in the top row while the approach using the maximum possible flux is in the bottom row. We point out the different behavior at both junctions.

First, we notice the first junction with one incoming and two outgoing roads, i.e. $x = 1$ in the figures. The approach using the maximum possible flux through the junction is zero up to time $t = 0.5$ because one of the outgoing roads (Road 3) reaches the maximal traffic density, hence the flux is zero (traffic jam) cf. Figure 1b. Our approach has nonzero traffic flow through this junction for $t \in [0, 0.5]$ because the numerical flux is nonzero between Road 1 and Road 2 allowing flows between these two roads. For times $t > 0.5$, the maximal traffic density is not attained on Road 3 and the traffic flow is nonzero through the junction in both cases, cf. Figure 1c. If we compare both approaches, we see completely different results on Roads 1 and 2 while the results on Road 3 are almost identical.



(a) $t = 0$. (b) $t = 0.25$. (c) $t = 0.75$.

Figure 1: Comparison of network with Road 1, Road 2 and Road 3. Top column – Our approach using numerical flux. Bottom column – Approach using maximum possible flux

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Now we focus on the second junction with two incoming and one outgoing road, i.e. $x = 0$ which is identified with $x = 2$ in the figures, due to periodicity. At first glance, there is no difference between the two approaches. Let's compare $\rho_1^{(R)}(0, 1)$, i.e. the limit from the right of traffic density on the outgoing Road 1 at $x = 0$ and $t = 1$. Our approach gives us $\rho_1^{(R)}(0, 1) \approx 0.4$ while the approach using the maximum possible flux gives us $\rho_1^{(R)}(0, 1) \approx 0.5$, which is the maximal traffic flow. The reason for this difference is that we do not have a defined right of way in our approach. Road 2 and Road 3 push too many cars into the junction congesting it slightly. The approach using the maximum possible flux takes into account the whole situation and selects the best solution for both roads. From a real point of view, this approach could be viewed as simulating the behavior of communicating autonomous vehicles which optimize the traffic situation globally, while our approach could be interpreted as simulating the behavior of human drivers without the right of way.

Conclusion

We have presented an overview of our paper [1] and demonstrated the numerical solution of macroscopic traffic flow models on network using the discontinuous Galerkin method. On individual roads, we use the Godunov numerical flux, while on junctions, we construct a new numerical flux based on the preferences of drivers. We compare our approach with the paper [3] by Čanić, Piccoli, Qiu and Ren, where Runge-Kutta methods are used along with a different choice of numerical fluxes at junctions. We discuss the differences between the two approaches, where that of [3] corresponds to single-lane roads with a strict enforcement of a priori traffic distribution, while the presented approach corresponds to having dedicated turning-lanes and/or flexibility of the drivers' preferences in extreme situations such as congestions. In future work, we would like to implement right of way rules (with regard to main and side roads) into the numerical flux and introduce true multi-lane roads with overtaking into the model.

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